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STABILITY OF THE EQUILIBRIUM OF A FLAT LAYER IN A MICROCONVECTION MODEL

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The stability of the equilibrium state of a flat layer bounded by rigid walls is studied using a microconvection model. The behavior of the complex decrement for long-wave perturbations has an asymptotic character. Calculations of the full spectral problem were performed for melted silicon. Unlike in the classical Oberbeck–Boussinesq model, the perturbations in the microconvection model are not monotonic. It is shown that for small Boussinesq parameters, the spectrum of this problem approximates the spectra of the corresponding problems for a heat-conducting viscous fluid or thermal gravitational convection when the Rayleigh number is finite.

1. Governing Equations. The Oberbeck–Boussinesq model describes adequately the thermal gravitational convection under earth's conditions. However, in very weak force fields, replacement of the continuity equation by the equation div $\boldsymbol{u} = 0$ leads to elimination of terms that can be as important as the term $-\beta\theta\boldsymbol{g}$, which expresses the contribution of the buoyancy force to the momentum equation. Pukhnachev [1] developed a model of microconvection in which the temperature dependence of density is given by

$$\rho = \rho_1(1+\beta\theta)^{-1},$$

where ρ_1 and β are positive constants. As in the classical Oberbeck–Boussinesq model, for small β, we obtain $\rho \approx \rho_1(1-\beta\theta)$.

Let $\boldsymbol{u}(x,y,z,t) = (u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t))$ be the velocity vector, and $p(x,y,z,t)$ be the fluid pressure. Following [1], we introduce new unknown variables:

$$\boldsymbol{w} = \boldsymbol{u} - \beta\chi\nabla\theta; \tag{1.1}$$

$$q = \rho_1^{-1}(p - \lambda \operatorname{div}\boldsymbol{u}) - \beta(\nu-\chi)\chi\Delta\theta. \tag{1.2}$$

Here χ is the thermal diffusivity, λ is the second viscosity coefficient, and $\nu = \mu/\rho_1$ is the kinematic viscosity. After some transformations [2], we obtain the following system of equations for the functions $\boldsymbol{w}$, q, and θ:

$$\boldsymbol{w}_t + \boldsymbol{w}\cdot\nabla\boldsymbol{w} + \beta\chi \operatorname{rot}\boldsymbol{w}\times\nabla\theta + \beta^2\chi^2 \operatorname{div}(\nabla\theta\otimes\nabla\theta - |\nabla\theta|^2 I)$$

$$= (1+\beta\theta)(-\nabla q + \nu\Delta\boldsymbol{w}) + \boldsymbol{g}; \tag{1.3}$$

$$\operatorname{div}\boldsymbol{w} = 0; \tag{1.4}$$

$$\theta_t + \boldsymbol{w}\cdot\nabla\theta + \beta\chi|\nabla\theta|^2 = (1+\beta\theta)\chi\Delta\theta. \tag{1.5}$$

Here $\boldsymbol{g}$ is the acceleration of gravity. The contribution of the dissipative function and pressure forces into the heat input equation (1.5) is assumed to be negligibly small.

At the initial time, it is necessary to specify the vector $\boldsymbol{w}$ and the temperature θ:

$$\boldsymbol{w}|_{t=0} = \boldsymbol{w}_1(\boldsymbol{x}) \equiv \boldsymbol{u}_1 - \beta\chi\nabla\theta_1, \quad \operatorname{div}\boldsymbol{w}_1 = 0, \quad \theta|_{t=0} = \theta_1(\boldsymbol{x}). \tag{1.6}$$

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The following conditions are satisfied on the rigid walls:

$$\boldsymbol{w} + \beta\chi\nabla\theta = 0; \tag{1.7}$$

$$\theta = \theta_w(\boldsymbol{x}, t) \quad \text{or} \quad k_1 \frac{\partial\theta}{\partial n} + b(\theta - \theta_g) = Q. \tag{1.8}$$

Equality (1.7) is the attachment condition ($\boldsymbol{u} = 0$) on the rigid wall; the first equality in (1.8) specifies the wall temperature and the second equality specifies heat transfer with the ambient medium (for $b = 0$, the heat flux).

Remark 1. For convection in a closed cavity Ω, from (1.4) and (1.7), we have the equality

$$\int\limits_{\Gamma} \frac{\partial\theta}{\partial n}\, d\Gamma = 0, \tag{1.9}$$

where Γ is the rigid wall surrounding the fluid. If we assume that $\rho = \rho(\theta)$, from the laws of conservation of mass and energy and from the attachment condition, we have

$$\int\limits_{\Gamma} V_\theta \frac{\partial\theta}{\partial n}\, d\Gamma = \int\limits_{\Omega} V_{\theta\theta} |\nabla\theta|^2\, d\Omega, \tag{1.10}$$

where $V = 1/\rho(\theta)$ is the specific volume. For model (1.3)–(1.5), $V = (1 + \beta\theta)/\rho_1$, and from (1.10), we obtain equality (1.9). Condition (1.9) [or more general (1.10)] is a necessary condition for the density to be independent of pressure.

We assume that l_* and θ_* are the characteristic length and temperature. We introduce dimensionless variables by the relations

$$\boldsymbol{x} \leftrightarrow l_*\boldsymbol{x}, \quad t \leftrightarrow l_*^2 t/\chi, \quad \boldsymbol{w} \leftrightarrow l_*^{-1}\chi\boldsymbol{w}, \quad \theta \leftrightarrow \theta_*\theta, \quad q \leftrightarrow q\nu\chi l_*^{-2}. \tag{1.11}$$

Then, system (1.3)–(1.5) is written as

$$\boldsymbol{w}_t + \boldsymbol{w}\nabla\boldsymbol{w} + \varepsilon\ \operatorname{rot}\boldsymbol{w} \times \nabla\theta + \varepsilon^2 \operatorname{div}(\nabla\theta \otimes \nabla\theta - |\nabla\theta|^2 I)$$

$$= (1 + \varepsilon\theta)(-\nabla\bar{q} + \Delta\boldsymbol{w})\mathrm{Pr} - \varepsilon\boldsymbol{\eta}(t)\mathrm{Pr}\,\theta; \tag{1.12}$$

$$\operatorname{div}\boldsymbol{w} = 0; \tag{1.13}$$

$$\theta_t + \boldsymbol{w}\cdot\nabla\theta + \varepsilon|\nabla\theta|^2 = (1 + \varepsilon\theta)\Delta\theta, \tag{1.14}$$

where $\mathrm{Pr} = \nu/\chi$ is the Prandtl number, $\varepsilon = \beta\theta_*$ is the Boussinesq parameter, and $\boldsymbol{\eta} = l_*^3\boldsymbol{g}(t)/(\nu\chi)$ is a vector microconvection parameter. In particular, if $\boldsymbol{g} = (0, 0, -g)$, then $\eta = l_*^3 g/(\nu\chi)$ is a microconvection parameter. For $\eta < 1$ [1], the Oberbeck–Boussinesq approximation is inadequate for describing convection. Boundary condition (1.7) becomes

$$\boldsymbol{w} + \varepsilon\nabla\theta = 0. \tag{1.15}$$

In Eq. (1.12), the analogue of the modified pressure is $\bar{q} = q - l_*^3\boldsymbol{g}(t)\cdot\boldsymbol{x}/(\nu\chi)$.

The parameter ε is included in system (1.12)–(1.15) in a regular manner (usually, its real value does not exceed 10^{-2}). Therefore, an analysis of Eqs. (1.12)–(1.15) leads to the following conclusions:

1. For moderate Prandtl numbers and $\varepsilon \to 0$, the microconvection system approximates the equations for a viscous heat-conducting fluid.

2. If $\mathrm{Pr} \gg 1$, in the limit we obtain the system of "creeping" motion:

$$\Delta\boldsymbol{w} - \nabla\bar{q} = \varepsilon\boldsymbol{\eta}(t)\theta, \qquad \operatorname{div}\boldsymbol{w} = 0, \tag{1.16}$$

$$\theta_t + \boldsymbol{w}\cdot\nabla\theta + \varepsilon|\nabla\theta|^2 = (1 + \varepsilon\theta)\Delta\theta.$$

3. If $\varepsilon\boldsymbol{\eta}(t) \to \mathbf{R}(t) \neq 0$ as $\varepsilon \to 0$, we obtain the Oberbeck–Boussinesq model [$\mathbf{R}(t)$ is the Rayleigh number vector]. We note that $\mathrm{Pr}\ \varepsilon\boldsymbol{\eta}(t) = \beta\theta_* l_*^3\boldsymbol{g}(t)/\chi^2 = \mathbf{Gr}$ ($\mathbf{Gr}$ is the Grashof number vector).

Series expansion of the solution in ε (or series expansion in Pr^{-1} in conclusion 2) in a zeroth approximation yields one of the models indicated above. For the problem (1.3)–(1.7), Pukhnachev [3] proved the existence of an analytical solution in ε in the Hölder classes. Microconvection was considered in a closed region Ω, and boundary condition (1.8) was brought to the form $\partial\theta/\partial n = Q/k_1$. Because in [3], the characteristic velocities and pressures are different from those in (1.11), the ultimate problem for $\varepsilon = 0$ cannot be physically interpreted.

Following [3], we can justify conclusions 1 and 2 mathematically. In the present paper, this is done using as an example a numerical solution of the full spectral problem that arises in studying the stability of fluid equilibrium in the microconvection model.

2. Equilibrium State. In the equilibrium state, $\boldsymbol{u} = 0$ and $\theta_t = p_t = 0$. Hence, from (1.1), it follows that

$$\boldsymbol{w}_0 = -\beta\chi\nabla\theta_0$$

(the subscript 0 refers to the equilibrium state), and, according to (1.4), the temperature is a harmonic function:

$$\Delta\theta_0 = 0. \tag{2.1}$$

Equation (1.5) is an identical equation, and Eq. (1.3) is equivalent to the equation

$$\nabla q_0 = \boldsymbol{g}/(1 + \beta\theta_0). \tag{2.2}$$

We note that by virtue of relations (1.2) and (2.1), $q_0 = p_0/\rho_1$. Therefore, the necessary equilibrium condition has the form $\boldsymbol{g}\cdot\operatorname{rot}\boldsymbol{g} = 0$. It holds for a constant vector of external forces, and it follows from (2.2) that

$$\nabla\theta_0 \times \boldsymbol{g} = 0. \tag{2.3}$$

If $\boldsymbol{g} = (0, 0, -g)$ ($g = \text{const} > 0$), Eq. (2.3) holds only for $\theta_0 = \theta_0(z)$. In this case, from (2.1), we have $\theta_0(z) = c_1 z + c_2$ ($c_1, c_2 = \text{const}$). In particular, the equilibrium state of a layer with rigid walls ($|z| = l$) on which constant temperatures θ_1 and θ_2 are maintained is described by the formulas

$$\boldsymbol{w}_0 = (0, 0, \beta\chi(\theta_2 - \theta_1)/(2l)), \qquad \theta_0 = (\theta_1 - \theta_2)z/(2l) + (\theta_1 + \theta_2)/2,$$
$$q_0 = -\frac{2lg}{\beta(\theta_1 - \theta_2)}\ln\Big(1 + \beta\,\frac{\theta_1 + \theta_2}{2} + \beta\,\frac{\theta_1 - \theta_2}{2l}\,z\Big) + c_3, \qquad c_3 = \text{const}. \tag{2.4}$$

Here, unlike in the classical case, the analog of pressure [the function $q_0(z)$] is distributed under a logarithmic rather than linear law. In addition, solution (2.4) satisfies system (1.16).

Remark 2. From (2.4) as $\beta \to 0$, we obtain

$$\boldsymbol{w}_0 = \boldsymbol{u}_0 = 0, \quad \theta_0 = (\theta_1 - \theta_2)z/(2l) + (\theta_1 + \theta_2)/2, \quad q_0 = c_4 - gz, \quad c_4 = \text{const}. \tag{2.5}$$

Because the pressure $p_0 = q_0\rho_1$, system (2.5) corresponds to the equilibrium state of a heat-conducting viscous fluid layer. This follows from the fact that according to the substitution (1.1) and (1.2), as $\beta \to 0$, system (1.3)–(1.5) approximates the Navier–Stokes equations for a heat-conducting viscous fluid.

Remark 3. If in the expression for $q_0(z)$ from (2.4) we retain terms of second-order smallness over β and use $\bar{p}_0(z)$ to denote the deviation of pressure from hydrostatic one, we obtain the equilibrium state in the Oberbeck–Boussinesq model (see [4, 5]):

$$\boldsymbol{w}_0 = \boldsymbol{u}_0 = 0, \quad \theta_0 = \frac{\theta_1 - \theta_2}{2l}\,z + \frac{\theta_1 + \theta_2}{2}, \quad \frac{d\bar{p}_0}{dz} = \rho_1 g\beta\theta_0(z). \tag{2.6}$$

3. Linearized Problem of Small Perturbations in the Microconvection Model. Let $\boldsymbol{w}(\boldsymbol{x}, t)$, $q(\boldsymbol{x}, t)$, and $\theta(\boldsymbol{x}, t)$ be the known main motion, $\tilde{\boldsymbol{w}}(\boldsymbol{x}, t) = \boldsymbol{w}(\boldsymbol{x}, t) + \boldsymbol{W}(\boldsymbol{x}, t)$, $\tilde{q}(\boldsymbol{x}, t) = q(\boldsymbol{x}, t) + Q(\boldsymbol{x}, t)$, and $\tilde{\theta}(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t) + T(\boldsymbol{x}, t)$ be perturbed motion. We assume that $\boldsymbol{W}$, Q, and T and their derivatives are small. Substituting $\tilde{\boldsymbol{w}}$, $\tilde{q}$, and $\tilde{\theta}$ into Eqs. (1.3)–(1.5), we obtain the following linear problem with respect to $\boldsymbol{W}$, Q, and T [2]:

$$\boldsymbol{W}_t + \boldsymbol{w}\nabla\boldsymbol{W} + \boldsymbol{W}\nabla\boldsymbol{w} + \beta\chi(\operatorname{rot}\boldsymbol{W} \times \nabla\theta + \operatorname{rot}\boldsymbol{w} \times \nabla T)$$
$$+ \beta^2\chi^2[\Delta\theta\nabla T + \Delta T\nabla\theta - \nabla\theta\nabla(\nabla T) - \nabla T\nabla(\nabla\theta)]$$
$$= (1 + \beta\theta)(-\nabla Q + \nu\Delta\boldsymbol{W}) + \beta T(-\nabla q + \nu\Delta\boldsymbol{w}); \tag{3.1}$$

$$\operatorname{div}\boldsymbol{W} = 0; \tag{3.2}$$

$$T_t + \boldsymbol{w}\cdot\nabla T + \boldsymbol{W}\cdot\nabla\theta + 2\beta\chi\nabla\theta\cdot\nabla T = (1 + \beta\theta)\chi\Delta T + \beta\chi T\Delta\theta. \tag{3.3}$$

We note that in (3.1), the expression for $\beta^2\chi^2$ is equal to $\mathrm{div}[\nabla\theta\otimes\nabla T+\nabla T\otimes\nabla\theta-2I\nabla\theta\cdot\nabla T]$ because $\nabla\theta\nabla(\nabla T)$ $+\nabla T\nabla(\nabla\theta)=\nabla(\nabla\theta\cdot\nabla T)$.

On the rigid walls, the conditions

$$\boldsymbol{W}+\beta\chi\nabla T=0,\qquad T=0 \tag{3.4}$$

or

$$\boldsymbol{W}+\beta\chi\nabla T=0,\qquad k_1\frac{\partial T}{\partial n}+bT=0$$

are satisfied.

System (3.1)–(3.3) is supplemented by the initial data

$$\boldsymbol{W}\Big|_{t=0}=\boldsymbol{W}_1(\boldsymbol{x}),\qquad \mathrm{div}\,\boldsymbol{W}_1(\boldsymbol{x})=0,\qquad T\Big|_{t=0}=T_1(\boldsymbol{x}). \tag{3.5}$$

We consider the problem (3.1)–(3.5) in the case of equilibrium in the layer with the rigid walls defined by formulas (2.4). We introduce the dimensionless variables $[\boldsymbol{W}=(U,V,W)]$:

$$\xi=x/(2l),\quad \eta=y/(2l),\quad \zeta=z/(2l),\quad \tau=\chi t/(4l^2),$$

$$U_1=2lU/\chi,\quad V_1=2lV/\chi,\quad W_1=2lW/\chi,\quad Q_1=4l^2Q/(\nu\chi),\quad T_1=T/(\mu(\theta_1-\theta_2)),$$

$$l_*=2l,\qquad \theta_*=\mu(\theta_1-\theta_2)$$

($\mu=1$ if $\theta_1>\theta_2$ and $\mu=-1$ if $\theta_1<\theta_2$). After substitution of these variables into (3.1)–(3.4), we have the following system (the subscript 1 is omitted):

$$\begin{gathered}
U_\tau-\varepsilon\mu W_\xi-\mu\varepsilon^2T_{\xi\zeta}=(1+\beta\theta_0)(-Q_\xi+\Delta U)\mathrm{Pr}\,,\\
V_\tau-\varepsilon\mu W_\eta-\mu\varepsilon^2T_{\eta\zeta}=(1+\beta\theta_0)(-Q_\eta+\Delta V)\mathrm{Pr}\,,\\
W_\tau-\varepsilon\mu W_\zeta+\mu\varepsilon^2(T_{\xi\xi}+T_{\eta\eta})=(1+\beta\theta_0)(-Q_\zeta+\Delta W)\mathrm{Pr}\,+\mathrm{Gr}T/(1+\beta\theta_0),\\
U_\xi+V_\eta+W_\zeta=0,\\
T_\tau+\varepsilon\mu T_\zeta+\mu W=(1+\beta\theta_0)\Delta T.
\end{gathered} \tag{3.6}$$

Here $\varepsilon=\beta|\theta_1-\theta_2|$ is the Boussinesq parameter, $\mathrm{Gr}=\mu\beta(\theta_1-\theta_2)(2l)^3g/\chi^2$ is the Grashof number, and $\theta_0(\zeta)=(\theta_1-\theta_2)\zeta+(\theta_1+\theta_2)/2$.

Boundary conditions (3.4) on the rigid walls ($\zeta=-1/2$, $\zeta=1/2$) take the form

$$U+\varepsilon T_\xi=0,\quad V+\varepsilon T_\eta=0,\quad W+\varepsilon T_\zeta=0,\quad T=0. \tag{3.7}$$

We seek a solution of boundary problem (3.6)–(3.7) in the form of normal waves

$$(U,V,W,Q,T)=(U(\zeta),V(\zeta),W(\zeta),Q(\zeta),T(\zeta))\exp\left(i(\alpha_1\xi+\alpha_2\eta-C\tau)\right). \tag{3.8}$$

Here α_1 and α_2 are dimensionless wavenumbers along the x and y axes, respectively, and C is the complex decrement that determines the variation of perturbations with time. If $C=C_r+iC_i$, the perturbations oscillate at a frequency C_r; the decay or build-up of the perturbations are determined by the sign of the real part of C_i.

Substituting (3.8) into (3.6), for $|\zeta|<1/2$, we obtain a spectral problem with respect to the parameter C for the system of ordinary differential equations:

$$-iCU-i\alpha_1\mu\varepsilon W-i\alpha_1\mu\varepsilon^2T'=(1+\beta\theta_0)[U''-(\alpha_1^2+\alpha_2^2)U-i\alpha_1Q]\mathrm{Pr}\,; \tag{3.9}$$

$$-iCV-i\alpha_2\mu\varepsilon W-i\alpha_2\mu\varepsilon^2T'=(1+\beta\theta_0)[V''-(\alpha_1^2+\alpha_2^2)V-i\alpha_2Q]\mathrm{Pr}\,; \tag{3.10}$$

$$-iCW-\mu\varepsilon W'-[\mu\varepsilon^2(\alpha_1^2+\alpha_2^2)+\mathrm{Gr}/(1+\beta\theta_0)]T=(1+\beta\theta_0)[W''-(\alpha_1^2+\alpha_2^2)W-Q']\mathrm{Pr}\,; \tag{3.11}$$

$$i\alpha_1U+i\alpha_2V+W'=0; \tag{3.12}$$

$$-iCT+\mu\varepsilon T'+\mu W=(1+\beta\theta_0)[T''-(\alpha_1^2+\alpha_2^2)T] \tag{3.13}$$

(primes denote differentiation with respect to ζ).

Boundary conditions (3.7) for $|\zeta| = 1/2$ have the form

$$U = 0, \qquad V = 0, \qquad W + \varepsilon T' = 0, \qquad T = 0. \tag{3.14}$$

Squire transformation can be applied to problem (3.9)–(3.14). Multiplying (3.9) by $i\alpha_1$ and (3.10) by $i\alpha_2$ and denoting $Z = i\alpha_1 U + i\alpha_2 V$, we obtain the problem

$$-iCZ + \mu\varepsilon k^2 W + \mu\varepsilon^2 k^2 T' = (1 + \beta\theta_0)(Z'' - k^2 Z + k^2 Q)\mathrm{Pr}\,; \tag{3.15}$$

$$-iCW - \mu\varepsilon W' - (\mu\varepsilon^2 k^2 + \mathrm{Gr}/(1 + \beta\theta_0))T = (1 + \beta\theta_0)(W'' - k^2 W - Q')\mathrm{Pr}\,; \tag{3.16}$$

$$Z + W' = 0; \tag{3.17}$$

$$-iCT + \mu\varepsilon T' + \mu W = (1 + \beta\theta_0)(T'' - k^2 T), \tag{3.18}$$

where $k = \sqrt{\alpha_1^2 + \alpha_2^2}$ is a modified wavenumber.

For $|\zeta| = 1/2$, we have

$$Z = 0, \qquad W + \varepsilon T' = 0, \qquad T = 0. \tag{3.19}$$

A necessary and sufficient condition for "rough" instability of the equilibrium state (3.4) (i. e., instability to a first approximation) is that $\mathrm{Im}\, C > 0$ for at least one eigenvalue.

Remark 4. For $C = 0$, system (3.15)–(3.18) reduces to one equation of the sixth order in temperature perturbations:

$$L^2(xLT - \varepsilon^2 T') + (k^2 \mathrm{R}/x^2)T = 0, \qquad T = T' = T'' = 0, \qquad x = 1 + \beta\theta_{1,2},$$

where $x = 1 + \beta\theta_0(\zeta)$ and $L = \varepsilon^2 d^2/dx^2 - k^2$. However, even in this case, we were unable to integrate the last equation explicitly and to find critical Rayleigh numbers R in explicit form.

Remark 5. Since $\mathrm{Gr} = \varepsilon\eta\mathrm{Pr}$ [$\eta = (2l)^3 g/(\nu\chi)$ is a microconvection parameter], for moderate Prandtl numbers, the boundary-value problem (3.15)–(3.19) approximates the problem of the stability of equilibrium (2.5) as $\varepsilon \to 0$ (see Remark 2). If $\mathrm{Gr} \to \mathrm{Gr}_0 > 0$ as $\varepsilon \to 0$, we arrive at the problem of the stability of equilibrium (2.6) in the Oberbeck–Boussinesq model.

4. Asymptotic Behavior of Long Waves. We consider the asymptotic behavior of amplitude equations as $k \to 0$.

Because k^2 appears everywhere in the system, we set

$$Z = Z_0 + k^2 Z_1 + \ldots, \qquad W = W_0 + k^2 W_1 + \ldots,$$

$$Q = Q_0 + k^2 Q_1 + \ldots, \qquad T = T_0 + k^2 T_1 + \ldots, \qquad C = C_0 + k^2 C_1 + \ldots\,.$$

In a zeroth approximation, substitution of these equations into (3.15)–(3.18) yields the system

$$\begin{gathered} -iC_0 Z_0 = (1 + \beta\theta_0)Z_0''\mathrm{Pr}\,, \\ -iC_0 W_0 - \mu\varepsilon W_0' - \mathrm{Gr}T_0/(1 + \beta\theta_0) = (1 + \beta\theta_0)(W_0'' - Q_0')\mathrm{Pr}\,, \\ Z_0 + W_0' = 0, \\ -iC_0 T_0 + \mu\varepsilon T_0' + \mu W_0 = (1 + \beta\theta_0)T_0''. \end{gathered} \tag{4.1}$$

The boundary conditions for Z_i, W_i, Q_i, and T_i $(i = 0, 1)$ coincide with (3.19).

We write the equation for Z_0 in the form $Z_0'' = -iC_0 Z_0/((1 + \beta\theta_0)\mathrm{Pr}\,)$. Multiplying it by the complex conjugate quantity Z_0^* and integrating it over the interval $[-1/2; 1/2]$, we have

$$\frac{iC_0}{\mathrm{Pr}} \int\limits_{-1/2}^{1/2} \frac{|Z_0|^2\, d\zeta}{1 + \beta\theta_0} = \int\limits_{-1/2}^{1/2} |Z_0'|^2\, d\zeta.$$

It follows that the quantity C_0 is imaginary ($C_0 = iC_{0i}$ for $C_{0i} < 0$). Hence, long-wave perturbations attenuate monotonically irrespective of the sign of the difference $\theta_1 - \theta_2$. The form of C_{0i} can easily be specified. Indeed, the substitution $x = 1 + \beta\theta_0(\zeta) = 1 + \beta(\theta_1 - \theta_2)\zeta + \beta(\theta_1 + \theta_2)/2$ leads to the equation $xZ_0'' + \mu_0 Z_0 = 0$, where $\mu_0 = iC_0/(\mathrm{Pr}\ \varepsilon^2)$; as was proved above, $\mu_0 > 0$. On the other hand, the last equation has a general solution

$$Z_0 = \sqrt{x}(h_1 J_1(2\sqrt{\mu_0 x}) + h_2 Y_1(2\sqrt{\mu_0 x})), \qquad h_1, h_2 = \mathrm{const},$$

where J_1 and Y_1 are Bessel functions of the first and second kinds. Because $Z_0(x_{1,2}) = 0$ ($x_{1,2} = 1 + \beta\theta_{1,2} > 0$), it follows that $\tau = 2\sqrt{\mu_0 x_1}$ is a root of the transcendental equation

$$J_1(\tau)Y_1(\lambda_0\tau) - J_1(\lambda_0\tau)Y_1(\tau) = 0, \qquad \lambda_0 = \sqrt{x_2/x_1}.$$

The last equation has a denumerable number of real roots τ_n [7]. Hence,

$$C_{0n} = -(\mathrm{Pr}\ \varepsilon^2\tau_n^2/(4x_1))i \equiv iC_{0i}. \tag{4.2}$$

We consider the system of the first approximation in k^2. Instead of (4.1), we have the system

$$\begin{gathered}
-i(C_0 Z_1 + C_1 Z_0) + \varepsilon W_0 + \mu\varepsilon^2 T_0' = (1 + \beta\theta_0)(Z_1'' - Z_0 + Q_0)\mathrm{Pr}\,,\\
-\, i(C_0 W_1 + C_1 W_0) - \mu\varepsilon W_1' - \mathrm{Gr}T_1/(1 + \beta\theta_0) - \mu\varepsilon^2 T_0 = (1 + \beta\theta_0)(W_1'' - W_0 - Q_1')\mathrm{Pr}\,,\\
Z_1 + W_1' = 0,\\
-i(C_1 T_0 + C_0 T_1) + \mu\varepsilon T_1' + \mu W_1 = (1 + \beta\theta_0)(T_1'' - T_0).
\end{gathered} \tag{4.3}$$

From (4.3) for Z_1, we obtain the boundary-value problem

$$Z_1'' + iC_0 Z_1/(\mathrm{Pr}\,(1 + \beta\theta_0)) = (-iC_1 Z_0 + \mu\varepsilon W_0 + \mu\varepsilon^2 T_0')/(\mathrm{Pr}\,(1 + \beta\theta_0)) + Z_0 - Q_0,$$

$$Z_1(\pm 1/2) = 0.$$

A necessary and sufficient condition for unique solvability of this problem is that the right side of the last equation be orthogonal to the solution of the homogeneous conjugate equation, i.e., Z_0^*. From this, we have

$$iC_1 = \Bigl[\int\limits_{-1/2}^{1/2} \Bigl(\frac{\mu\varepsilon W_0 + \mu\varepsilon^2 T_0'}{(1 + \beta\theta_0)\mathrm{Pr}} - Q_0 + Z_0\Bigr) Z_0^*\, d\zeta\Bigr] \Bigl(\int\limits_{-1/2}^{1/2} \frac{|Z_0|^2}{1 + \beta\theta_0}\, d\zeta\Bigr)^{-1}. \tag{4.4}$$

It can be shown that iC_1 is a real number.

5. Numerical Solution of the Eigenvalue Problem. To find a numerical solution of the problem using the orthogonalization method [8], we reduce system (3.15)–(3.18) to the form and $\boldsymbol{y}' = A\boldsymbol{y}$, where $\boldsymbol{y}(\xi)$ is the vector of the unknown quantities and $A(\xi)$ is a coefficient matrix ($0 \leqslant \xi \leqslant 1$). We substitute

$$\xi = \zeta + 1/2, \quad y_1 = Z, \quad y_2 = Z', \quad y_3 = Z'', \quad y_4 = W, \quad y_5 = T, \quad y_6 = T'. \tag{5.1}$$

Excluding Q from (3.15) and (3.16), we obtain the following system of equations:

$$y_1' = y_2, \qquad y_2' = y_3, \qquad y_4' = -y_1, \qquad y_5' = y_6,$$

$$y_3' = \frac{\varepsilon Ci}{(1 + \beta\theta_0)^2\mathrm{Pr}}\, y_1 + \Bigl(2k^2 - \frac{Ci}{(1 + \beta\theta_0)\mathrm{Pr}}\Bigr) y_2$$

$$+ \Bigl(k^4 - \frac{k^2 Ci}{(1 + \beta\theta_0)\mathrm{Pr}} + \frac{\varepsilon k^2(\varepsilon - \mu)}{\mathrm{Pr}\,(1 + \beta\theta_0)^2}\Bigr) y_4 - \frac{\mu\varepsilon^2 k^2 Ci + k^2\mathrm{Gr}}{(1 + \beta\theta_0)^2\mathrm{Pr}}\, y_5 + \frac{\varepsilon^3 k^2(1 - \mu)}{\mathrm{Pr}\,(1 + \beta\theta_0)^2}\, y_6,$$

$$y_6' = \frac{\mu}{1 + \beta\theta_0} y_4 + \Bigl(k^2 - \frac{Ci}{1 + \beta\theta_0}\Bigr) y_5 + \frac{\varepsilon\mu}{1 + \beta\theta_0}\, y_6.$$

Here $\theta_0 = \theta_2 + (\theta_1 - \theta_2)\xi$. By virtue of substitution (5.1), boundary conditions (3.19) take the form $y_1 = 0$, $y_4 + \varepsilon y_6 = 0$, and $y_5 = 0$ for $\xi = 0$ and $\xi = 1$.

Thus, we solve the system $\boldsymbol{y}' = A(\xi)\boldsymbol{y}$ with the boundary conditions $B\boldsymbol{y}(0) = 0$ and $D\boldsymbol{y}(1) = 0$ for $\xi = 0$ and 1, respectively. The 6×6 matrix A has the following elements:

$$a_{11} = a_{13} = a_{14} = a_{15} = a_{16} = 0, \qquad a_{12} = 1,$$

$$a_{21} = a_{22} = a_{24} = a_{25} = a_{26} = 0, \qquad a_{23} = 1,$$

$$a_{31} = \frac{\varepsilon Ci}{(1+\beta\theta_0)^2 \mathrm{Pr}}, \quad a_{32} = 2k^2 - \frac{Ci}{(1+\beta\theta_0)\mathrm{Pr}}, \quad a_{33} = 0,$$

$$a_{34} = k^4 - \frac{k^2 Ci}{(1+\beta\theta_0)\mathrm{Pr}} + \frac{\varepsilon k^2(\varepsilon-\mu)}{\mathrm{Pr}\,(1+\beta\theta_0)^2}, \quad a_{35} = -\frac{\mu\varepsilon^2 k^2 Ci + k^2 \mathrm{Gr}}{(1+\beta\theta_0)^2 \mathrm{Pr}}, \quad a_{36} = \frac{\varepsilon^3 k^2 (1-\mu)}{\mathrm{Pr}\,(1+\beta\theta_0)^2},$$

$$a_{41} = -1, \qquad a_{42} = a_{43} = a_{44} = a_{45} = a_{46} = 0,$$

$$a_{51} = a_{52} = a_{53} = a_{54} = a_{55} = 0, \qquad a_{56} = 1,$$

$$a_{61} = a_{62} = a_{63} = 0, \quad a_{64} = \frac{\mu}{1+\beta\theta_0}, \quad a_{65} = k^2 - \frac{Ci}{1+\beta\theta_0}, \quad a_{66} = \frac{\varepsilon\mu}{1+\beta\theta_0}.$$

The 3×6 matrices B and D coincide and their elements have the values

$$b_{11} = d_{11} = b_{24} = d_{24} = b_{35} = d_{35} = 1, \qquad b_{26} = d_{26} = \varepsilon.$$

The other elements of both matrices are equal to zero.

We seek a solution in the form

$$\boldsymbol{y} = \sum_{j=1}^{3} p_j \boldsymbol{y}^j, \tag{5.2}$$

where the coefficients p_j are determined from the system $D\boldsymbol{y}(1) = 0$, and $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, and $\boldsymbol{y}^3$ are linearly independent vectors such that

$$\boldsymbol{y}^1(0) = (0,0,0,-\varepsilon,0,1), \qquad \boldsymbol{y}^2(0) = (0,1,0,0,0,0), \qquad \boldsymbol{y}^3(0) = (0,0,1,0,0,0).$$

To determine the eigenvalue C, it is necessary to choose two initial approximations C_0 and C_1 from conditions (4.2) and (4.4). In the leftmost region, we integrate the equations for $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, and $\boldsymbol{y}^3$ with a specified step size in ξ. We orthogonalize the vectors obtained in the right region. In the next region, we integrate only those solutions for which the initial data are vectors obtained by orthogonalization. The solutions on the right end of the second region are orthogonalized until the point $\xi = 1$ is reached. For integration, we use the forth-order Runge–Kutta–Mercenne method with an automatic choice of an integration step. Since for each of the vectors $\boldsymbol{y}^j$ there may be an individual integration step, we retain the smallest of the three values obtained by automatic choice of the step. Reaching the right side of the integration region (point $\xi = 1$), we have a system of three equations $D\boldsymbol{y}(1) = 0$ for three unknowns p_j, where $\boldsymbol{y}$ has the form of (5.2). The determinant of the system composed of the coefficients y_i^j ($j = 1,2,3$, $i = 1,\ldots,6$) is written as a characteristic polynomial $F(C)$. A necessary and sufficient conditions for the existence of a nontrivial solution of the system $D\boldsymbol{y}(1) = 0$ is that the determinant of the system [in this case, $F(C)$] be equal to zero. Thus, the problem reduces to solution of the nonlinear equation $F(C) = 0$. The equation is solved by the secant method, using C_0 and C_1 as the initial approximations. The root of the equation $F(C) = 0$ is the desired eigenvalue for a specified wavenumber k. We consider long-wave perturbations, i.e., $k \to 0$. Moving along k from the value $k = 10^{-5}$, we find the dependence $C(k)$. From the sign of the imaginary components C obtained in each step over k, we determine the stability intervals.

We studied the stability of a layer of melted silicon with rigid walls for the following parameters: $\nu = 2.65 \cdot 10^{-3}$ cm^2/sec, $\chi = 0.49$ cm^2/sec, $\beta = 0.75 \cdot 10^{-5}\,{}^\circ\mathrm{C}^{-1}$, and $\mathrm{Pr} = 5.41 \cdot 10^{-3}$. Calculations were performed for absolute values of the temperature difference on the walls: $|\theta_1 - \theta_2| = 10$ and 1000°C. This means variation in the dimensionless parameter $\varepsilon = \beta|\theta_1 - \theta_2|$. The linear dimension of the layer was chosen such that the inequality $(2l)^3 g/(\nu\chi) < 1$, which is a criterion for the validity of the microconvection model (see [1, 9]), was satisfied. The smallness of the parameter $\eta = (2l)^3 g/(\nu\chi)$ can be reached by increasing the length scale or the acceleration of gravity g [for example, under zero gravity with $g \approx (10^{-2}$–$10^{-3})g_0$, where $g_0 = 981$ cm/sec^2 is the acceleration of gravity near the earth]. In the calculations, $g \approx 10^{-3} g_0$, i. e., $2l < 0.11$ cm. For the indicated values of l, β, χ, and ν, we determined the dependence of the parameters $C_i = \operatorname{Im} C$ and $C_r = \operatorname{Re} C$ on the wavenumber k.

Figure 1 shows curves of $C_i(k)$ for $\varepsilon = 7.5 \cdot 10^{-5}$ ($|\theta_1 - \theta_2| = 10^\circ$C) and Rayleigh number $\mathrm{R} = 4.21 \cdot 10^{-4}$. The dashed curve corresponds to heating from above ($\theta_1 > \theta_2$) and the solid curve to heating from below ($\theta_1 < \theta_2$). (Hereinafter, the curves corresponding to the case where the fluid is heated from above are denoted by C_i^- and those corresponding to heating from below are denoted by C_i^+.)

Figure 2 shows a curve of $C_r(k)$ for the same values of ε and R as in Fig. 1. Since the values of $C_r^+(k)$ and $C_r^-(k)$ differ from each other by not more than 10^{-6}, the corresponding curves in Fig. 2 coincide.

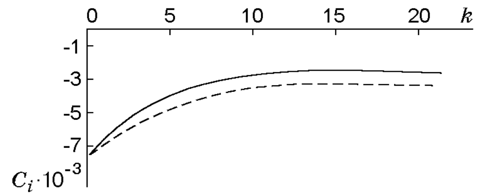

Fig. 1

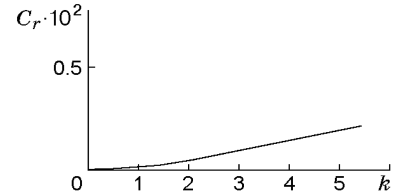

Fig. 2

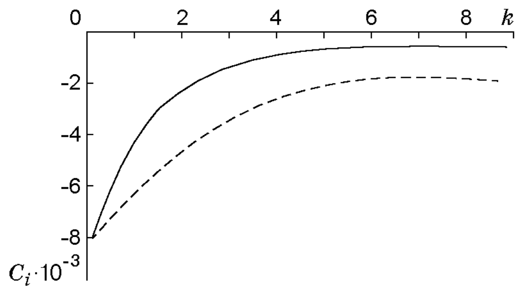

Fig. 3

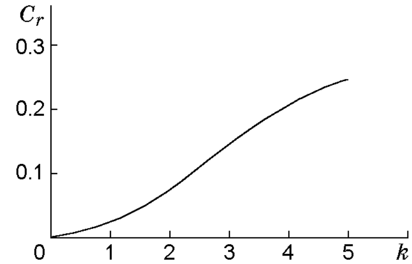

Fig. 4

Figure 3 shows curves of $C_i^+(k)$ and $C_i^-(k)$ (solid and dashed curves, respectively) for $\varepsilon = 7.5 \cdot 10^{-3}$ ($|\theta_1 - \theta_2| = 1000°$C) and $\mathrm{R} = 4.21 \cdot 10^{-2}$. Figure 4 shows a curve of $C_r(k)$ for the same values of ε and R as in Fig. 3.

We note that with increase in ε (i.e., the temperature difference on the walls), the curves of $C_i(k)$ increases more rapidly from the value of C_{0i}. All curves of $C_i^-(k)$ increase more slowly in comparison with the corresponding curves of $C_i^+(k)$. With increase in ε, the modulus of the difference $|C_i^+ - C_i^-|$ increases. For any k, all values of $C_i < 0$, i.e., the equilibrium state is stable.

All curves of $C_r(k)$ grow only slightly with increase in k ($10^{-7} \leqslant k \leqslant 1$). For all values of ε, the following conditions are satisfied: 1) $C_r > 0$ for all k; moreover, the values of C_r are close to zero ($C_r \approx 10^{-12}$) up to $k = 0.05$; 2) C_r practically does not change at $k \geqslant 5$: $|C_r(5) - C_r(20)| \leqslant 10^{-12}$; 3) all values of C_r^- lie below the corresponding values of C_r^+: $|C_r^+(k) - C_r^-(k)| < 10^{-6}$.

The stability of equilibrium (2.4) for melted silicon is not unexpected because $\mathrm{Pr} = 5.41 \cdot 10^{-3}$ (see Remark 5). If we set $\varepsilon = 0$ in (3.15)–(3.19), we obtain the problem of the stability of equilibrium (2.5) for a heat-conducting viscous fluid:

$$-iCZ = (Z'' - k^2 Z + k^2 Q)\mathrm{Pr}\,, \quad -iCW = (W'' - k^2 W - Q')\mathrm{Pr}\,, \quad -iCT + \mu W = T'' - k^2 T,$$

$$Z + W' = 0, \quad -1/2 < \xi < 1/2, \qquad Z = W = T = 0, \quad \xi = \pm 1/2.$$

This spectral problem is easy to solve: we first determine Z and W and then find the temperature perturbations. Although explicit expressions are not given here, we note that the following integral identity holds:

$$\Big(k^2 - \frac{iC}{\mathrm{Pr}}\Big) \int\limits_{-1/2}^{1/2} (k^2|W|^2 + |Z|^2)\, d\xi + \int\limits_{-1/2}^{1/2} (k^2|W'|^2 + |Z'|^2)\, d\xi = 0.$$

Hence it follows that $-iC < 0$ is a real number. In other words, the equilibrium state (2.5), which is limiting for (2.4) as $\beta \to 0$, is always stable. It can be shown that the complex decrement is a solution of one of the equations

$$x \tan x = -k \tanh\,(k/2), \qquad (1/x) \tan x = (1/k) \tanh\,(k/2),$$

where $x = (iC/\mathrm{Pr} - k^2)^{1/2}/2$. The last equations have a denumerable number of real solutions.

As is known, the linearized problem of the convective unstability of an immovable fluid in the Oberbeck–Boussinesq model is a self-conjugate problem (in the case of heating from below) [4]; therefore, the real part of the eigenvalue C_r is equal to zero. The perturbations attenuate or intensify monotonically, and the resulting motion is steady. The equilibrium state (2.6) of a horizontal fluid layer with thickness of $2l$ and the temperature

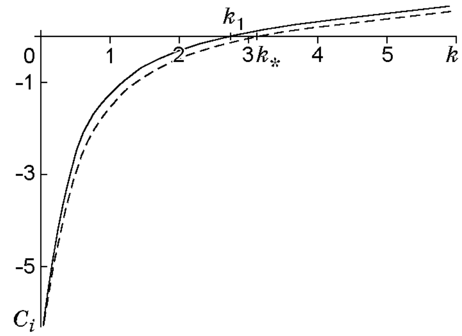

Fig. 5

gradient directed below $[(\theta_2 - \theta_1)/(2l) > 0]$ becomes unstable, if $\mathrm{R} = g\beta(\theta_2 - \theta_1)(2l)^3/(\nu\chi) > \mathrm{R}_* = 1708$, and the corresponding dimensionless value of the wavenumber is $k_* = 3.12$.

It is of interest (see Remarks 3 and 5) to compare the classical result with results of numerical solution of the spectral problem (3.15)–(3.19) when the Rayleigh number $\mathrm{R} = \varepsilon\eta$ is finite for $\varepsilon \ll 1$. Calculations were performed for melted silicon for the same values of physical parameters and for $\theta_2 - \theta_1 = 1000°$C. With increase in η, the curve of $C_i(k)$ approaches the axis $C_i = 0$ and intersects this axis for the first time at $k = k_1 = 2.84 < k_*$, when $\eta_1 = 225,193.33$. In this case, $\mathrm{R}_1 = \varepsilon\eta_1 = 1688.95 < \mathrm{R}_*$ and the layer thickness is $2l_1 = 6.68$ cm for $g = 10^{-3}g_0$. The solid curve in Fig. 5 shows a curve of $C_i(k)$ for the microconvection model, and the dashed curve shows the same curve for the Oberbeck–Boussinesq model. Thus, in the microconvection model, instability of the equilibrium state is observed for smaller wavenumbers. Obviously, this is due to the larger mobility (compressibility) of the fluid in this case. The values of C_r for $\mathrm{R} > 10^3$ for all k are close to 10^{-12}, and the spectral problem (3.15)–(3.19) becomes more "self-conjugate." With decrease in the Boussinesq parameter ε, the critical values of the Rayleigh number and the wavenumber increase. Thus, for $\varepsilon = 0.75 \cdot 10^{-4}$ and $\theta_2 - \theta_1 = 10°$C we have $k_1 = 2.99$ and $\mathrm{R}_1 = 1694.54$, which agrees with Remark 5.

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